Journal of Pure and Applied Algebra 15 (1979) 53–73

SEMI-TOPOLOGICAL FUNCTORS I

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Communicated by G.M. Kelly
Received 20 September 1977

1. Introduction

The aim of this paper is to give an appropriate description of "nice concrete functors" such that "algebraic" as well as "topological" functors and their compositions are included. Moreover, one should be able to prove all common properties of these functors. The means to get such a description is the concept of relative factorizations. In [11] Herrlich proved the existence of two kinds of factorizations of morphisms $X \to PA$ where $P: \mathbf{A} \to \mathbf{X}$ is a given functor. This concept was generalized in [29] and extended from morphisms to cones over a fixed diagram category in [31]. In the case of an identical functor this is nothing but the image factorization of morphisms resp. cones in a category (cp. [7, 24] and references there).

A special kind of these relative factorizations of cones was implicitly used to introduce the different notions of topological functors which are based on a categorical treatment of initiality and finality: Cp. Brümmer [3], Hoffmann [15], Wischnewsky [37, 38], Wyler [40, 41] (earlier or related investigations can be found in [1, 9, 20, 22, 25, 28]; mostly based on Wyler's papers are [4, 6, 23, 27, 29]). The first explicit use of cone factorizations to introduce a generalized type of topological functors appeared in Herrlich's paper [12]. On the other hand, Herrlich [13] has also used cone factorizations to describe an "algebraic" type of functors. In both papers all cones are allowed to be large so that any smallness conditions are superfluous. Extending his relative cone factorization to arbitrary diagram categories the author introduced the notion of an (orthogonal)**M**-functor [32, 33] which includes both "algebraic" and "topological" functors. A related notion was introduced by Hong [19].

But there are some reasons to generalize this notion once more. Topological functors are self-dual [15, 25], i.e. they can be equivalently described by initiality or finality. Hoffmann [16] proved a duality theorem even for Herrlich's generalized topological functors using so called semi-identifying lifts (cp. [5, 10, 30, 42] for related notions). Semi-topological functors as introduced in this paper generalize **M**-functors just so far such that they allow both an initial and a final characterization. This is the only justification for the name "semi-topological".

Semi-topological functors arise everywhere: All types of topological functors treated in the papers mentioned before are semi-topological; monadic functors over **Ens** and regular functors [13] are semi-topological; wellbounded categories [39] and locally presentable categories [8] admit a semi-topological functor into some power of **Ens**. Semi-topological functors have all the nice lifting properties one can expect. (In this paper we only prove "internal" lifting properties; for external ones see [36]). Furthermore they can be externally characterized in the language of 2-categories; this will be outlined in [35]. The main results of this paper are the generalized "Duality Theorem" 3.1 (already announced in [34]), the characterization Theorem 6.3 (which generalizes corresponding results of Herrlich [12] and Hoffmann [17]) and the representation theorem 8.3, which was proved at first "directly" by M.B. Wischnewsky and the author during the meeting on "Kategorien" in Oberwolfach 1977. The essentially new proof 3.2 of faithfulness of semi-topological functors was inspired by discussions with R. Börger. The method used there leads also to the useful Corollary 6.4 and will be worked out in [2]. Besides M.B. Wischnewsky and R. Börger I am indebted to D. Pumplün and G. Greve for some useful discussions on the subject of this paper.

2. Semi-initial and semi-final liftings

Let $P: \mathbf{A} \to \mathbf{X}$ be a functor. A *P-cone* is a triple (X, ξ, D), where X is an object in $\mathbf{X}$, D is a diagram in $\mathbf{A}$ (i.e., a functor $D: \mathbf{D} \to \mathbf{A}$, $\mathbf{D}$ any category) and $\xi: \Delta X \to P \circ D$ is a natural transformation ($\Delta = \Delta_{\mathbf{D}}$ always denotes the canonical functor into a functor category). Often, for brevity, we call ξ a P-cone. *Cone*(P) denotes the "class" of all P-cones. Reversing the direction of ξ one gets the notion of a *P-co-cone* and the "class" *Co-Cone*(P). In case $\mathbf{D} = \mathbf{1}$ (one point category), P-cones can be presented as pairs (x, A), where A is an object of $\mathbf{A}$ and $x: X \to PA$ is a morphism in $\mathbf{X}$. They form the class *Mor*(P) of so called *P-morphisms*. Dually one gets the class *Co-Mor*(P) of *P-co-morphisms*. An object of $\mathbf{X}$ can be regarded as a P-cone or a P-co-cone over the empty category $\mathbf{0}$.

2.1. Definition. Let $\xi: \Delta X \to P \circ D$ be a P-cone. A *P-semi-initial lifting* of ξ consists of a P-morphism $e: X \to PA$ and an $\mathbf{A}$-cone $\alpha: \Delta A \to D$ with

(*) $\qquad (P \circ \alpha)(\Delta e) = \xi,$

such that condition (SI) holds:

(SI) $\qquad$ For all P-co-morphisms $y: PB \to X$ and $\mathbf{A}$-cones $\beta: \Delta B \to D$ with $\xi(\Delta y) = P \circ \beta$ there is a unique morphism $b: B \to A$ with $ey = Pb$ and $\alpha(\Delta b) = \beta$.

Often we call (*) a *P-semi-initial factorization* of ξ. If e can be chosen as an isomorphism (identity morphism), α is called a (*proper*) *P-initial lifting* of ξ.

$$\begin{tikzcd}
 & & \Delta PA \arrow[dd, "P\circ\alpha"] \\
\Delta PB \arrow[urr, dashed, "\Delta Pb"] \arrow[r, "\Delta y"] \arrow[drr, "P\circ\beta"'] & \Delta X \arrow[ur, "\Delta e"] \arrow[dr, "\xi"] & \\
 & & P\circ D
\end{tikzcd}$$

Fig. 1.

A factorization (*) of ξ is called *rigid*, if condition (R) holds:

(R) For all endomorphisms $t: A \to A$ with $(Pt)\ e = e$ and $\alpha(\Delta t) = \alpha$ one has $t = A$.

2.2. Remarks. (1) Condition (SI) is fulfilled, if in (*) $\alpha: \Delta A \to D$ can be chosen as a *P-initial* cone, i.e. α is a P-initial lifting of $P \circ \alpha: \Delta PA \to P \circ D$.

(2) Condition (R) is fulfilled, if in (*) $e: X \to PA$ can be chosen as a P-epimorphism, i.e. for all $f, g: A \to C$ the equation $(Pf)e = (Pg)e$ implies $f = g$.

(3) For every functor, all P-morphisms $x: X \to PA$ have a trivial rigid P-semi-initial factorization: $x = (PA)x$.

To illustrate the notions in 2.1 we restrict ourselves here to two characteristic examples.

2.3. Examples. (1) Let P: **Top** $\to$ **Ens** be the forgetful functor of the category of topological spaces and let $\xi: \Delta X \to P \circ D$ be a P-cone. Then there is a coarsest topology on the set X making all mappings $X \to Dd$, $d \in Ob$ **D**, continuous. In this way one gets a space A and a P-initial cone $\alpha: \Delta A \to D$ with $P \circ \alpha = \xi$. In particular,

$$(P \circ \alpha)(\Delta X) = \xi$$

is a rigid P-semi-initial factorization.

(2) Let P: **Grp** $\to$ **Ens** the forgetful functor of the category of groups and let $x: X \to A$ be a mapping from a set to a group. Then its factorization over the subgroup $\langle im\ x \rangle$ of A generated by $im\ x$ yields a rigid P-semi-initial factorization of x. In case of a small P-cone ξ: $\Delta X \to P \circ D$ (D: **D** $\to$ **Grp**, **D** small) it suffices to factorize the induced mapping $x: X \to \prod_d Dd$. If ξ is an arbitrary P-cone one first has to factorize all mappings $\xi d: X \to Dd$ and then to choose a representative set $\{\xi' i: X \to \langle im\ (\xi d_i) \rangle | i \in \mathbf{I}\}$, which reduces ξ to a small P-cone ξ'. In general, this procedure is treated in 7.5.

The usefulness of semi-initial liftings is demonstrated already by the following lemma.

2.4. Limit Lemma. *Let* $(*)$ *be a rigid P-semi-initial lifting of a limit-cone* ξ, *then* α *is a limit-cone and* e *an isomorphism of* $\mathbf{X}$.

Proof. The limit property of ξ gives us a unique morphism $y: PA \to X$ with $\xi(\Delta y) = P \circ \alpha$. Necessarily we have $ye = X$ and, by (SI), there is a morphism $t: A \to A$ with $Pt = ey$ and $\alpha(\Delta t) = \alpha$. Because of (R), t must be an identity morphism, and therefore e is an isomorphism. Hence α is a P-initial lifting of a limit-cone and so a limit-cone itself (cp. [12, 31]).

2.5. Definition. Let $\xi: P \circ D \to \Delta X$ be a P-co-cone. A *P-semi-final lifting* of ξ consists of a P-morphism $e: X \to PA$ and an $\mathbf{A}$-co-cone $\alpha: D \to \Delta A$ with

$$(**) \qquad (\Delta e)\xi = P \circ \alpha,$$

such that condition (SF) holds:

(SF) For all P-morphisms $y: X \to PB$ and $\mathbf{A}$-co-cones $\beta: D \to \Delta B$ with $(\Delta y)\ \xi = P \circ \beta$ there is a unique morphism $t: A \to B$ with $(Pt)e = y$ and $(\Delta t)\alpha = \beta$.

$$\begin{tikzcd}
 & & \Delta PA \arrow[dd, dashed, "\Delta Pt"] \\
P \circ D \arrow[r] \arrow[urr, "P \circ \alpha"] \arrow[drr, "P \circ \beta"'] & \Delta X \arrow[ur, "\Delta e"] \arrow[dr, "\Delta y"] & \\
 & & \Delta PB
\end{tikzcd}$$

Fig. 2.

Often we call $(**)$ a *P-semi-final prolongation* of ξ. If e can be chosen as an isomorphism (identity morphism), α is called a (proper) *P-final lifting* of ξ.

2.6. Remarks. (1) "Final" is the dual notion of "initial", but "semi-final" is not dual to "semi-initial". The dual notions are "*co-semi-initial*" and "*co-semi-final*".

(2) Obviously, a P-semi-final prolongation of an P-co-cone is uniquely determined up to a canonical isomorphism, whereas a P-cone can have many different rigid P-semi-initial factorizations.

(3) A P-morphism $e: X \to PA$ is called a *P-quotient*, if it appears in some P-semi-final prolongation, i.e. there is a P-co-cone $\xi: P \circ D \to \Delta X$ and a cone $\alpha: D \to \Delta A$ with $(**)$ and (SF). Let us consider the following subclasses of $Mor(P)$:

$Iso(P) := \{(e, A) \mid e \text{ is an isomorphism of } \mathbf{X}\}$,
$Quot(P) := \{(e, A) \mid (e, A) \text{ is a } P\text{-quotient}\}$,
$Epi(P) := \{(e, A) \mid (e, A) \text{ is a } P\text{-epimorphism}\}$ (cp. 2.2(2)).

For P faithful we then have the inclusions

$$Iso(P) \subset Quot(P) \subset Epi(P).$$

Again we consider our two basic examples 2.3:

2.7. Examples. (1) For $P\colon \mathbf{Top} \to \mathbf{Ens}$ let $\xi\colon P \circ D \to \Delta X$ be a P-co-cone. Then there is a finest topology on X making all mappings $Dd \to X$, $d \in Ob\mathbf{D}$, continuous. So we have a proper P-final lifting of ξ which, in particular, yields a P-semi-final prolongation of ξ.

(2) In case $\mathbf{Grp} \to \mathbf{Ens}$ form the free group FX on X and consider the smallest congruence relation $\sim$ on FX such that all mappings $Dd \to FX/\sim$ become homomorphisms. Then the canonical map $X \to FX/\sim$ gives a P-semi-final prolongation of ξ.

2.8. Co-Limit Lemma. *Let (**) be a P-semi-final lifting of a co-limit-co-cone ξ, then so is α. In general, e is not an isomorphism in* $\mathbf{X}$.

Proof. Straightforward, using property (SF). Co-limits are not necessarily preserved by P, hence e is not an isomorphism in general.

The fundamental connection between semi-initial and semi-final liftings is given by the following lemma, which we shall use later.

$$\begin{tikzcd}
\Delta X \arrow[r, "\Delta p"] \arrow[d, "\Delta x"'] & \Delta PA \arrow[dd, "P\circ\alpha"] \\
\Delta Y \arrow[d, "\Delta q"'] & \\
\Delta PB \arrow[r, "P\circ\mu"] & P\circ D
\end{tikzcd}$$

Fig. 3.

2.9. Diagonal-Lemma. *In the commutative diagram of Fig.* 3 *let* (p, A) *be a P-quotient* (*cp.* 2.6(3)) *and let* $(P \circ \mu)(\Delta q) =: \xi$ *be a P-semi-initial factorization. Then there exists a unique morphism* $t\colon A \to B$ *with* $(Pt)p = qx$ *and* $\mu(\Delta t) = \alpha$.

Proof. By definition, there exists a P-co-cone $\zeta\colon P \circ C \to \Delta X$ and an $\mathbf{A}$-co-cone $\gamma\colon C \to \Delta A$ with $C\colon \mathbf{C} \to \mathbf{A}$ such that $(\Delta p)\zeta = P \circ \gamma$ is a P-semi-final prolongation. Because of (SI), for all $c \in Ob\mathbf{C}$ there is a unique $\beta c\colon Cc \to B$ with $qx\,(\zeta c) = P\beta c$ and $\mu(\Delta\beta c) = \alpha(\Delta\gamma c)$. So we get an $\mathbf{A}$-co-cone $\beta\colon C \to \Delta B$ with $(\Delta q)(\Delta x)\zeta = P \circ \beta$ and therefore, because of (SF), a unique morphism $t\colon A \to B$ with $(Pt)p = qx$ and $(\Delta t)\gamma = \beta$. Now the equation $\mu(\Delta t) = \alpha$ and the uniqueness of t is proved canonically.

3. The generalized "Duality Theorem"

Fundamental for the definition of semi-topological functors is the following theorem generalizing the well known "Duality Theorem" for topological functors (cp. [15, 25]; see 4.3 below).

3.1. Theorem. *Let $P: \mathbf{A} \to \mathbf{X}$ be a functor and let $\mathbf{Q} \subset Mor(P)$ be a subclass. Then the following assertions are equivalent*:

(i) *Every P-cone ξ has a rigid P-semi-initial lifting* (*) *with* $(e, A) \in \mathbf{Q}$.
(ii) *Every P-co-cone ξ has a P-semi-final lifting* (**) *with* $(e, A) \in \mathbf{Q}$.

In order to prove this theorem we need first the following lemma.

3.2. Lemma. *Each of the conditions* (i) *and* (ii) *implies that P is faithful.*

Proof. (i). Let $f, g: A \to B$ be $\mathbf{A}$-morphisms with $Pf = Pg$ and $f \neq g$. Consider the $\mathbf{I}$-indexed discrete *P-cone* $(PA, x_i, B_i;\, i \in \mathbf{I})$ with $\mathbf{I} := Mor(\mathbf{A})$ and

$$B_i := B, \qquad x_i := Pf: PA \to PB_i \quad \text{for all } i \in \mathbf{I}$$

and form a P-semi-initial factorization

$$(Pa_i)e = x_i, \qquad a_i: C \to B_i \quad \text{for all } i \in \mathbf{I}.$$

The class $\mathbf{K} := \{h: A \to C \mid a_i h \in \{f, g\} \text{ for all } i \in \mathbf{I}\}$ is not empty: In Fig. 4, take all b_i to be f and apply (SI). Hence there exists a surjection $\sigma: \mathbf{I} \to \mathbf{K}$ (take $\sigma/\mathbf{K}$ to be $\mathrm{id}_{\mathbf{K}}$ and $\sigma/\mathbf{I} - \mathbf{K}$ to be constant) and we can define

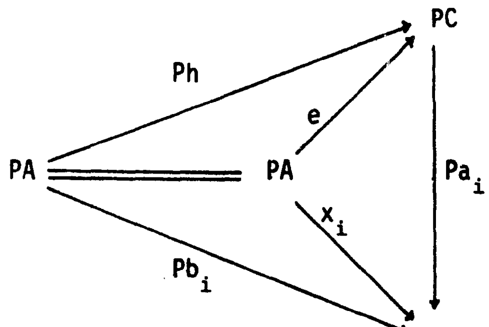

Fig. 4.

$$b_i := \begin{cases} f & \text{in case } a_i\sigma(i) = g, \\ g & \text{in case } a_i\sigma(i) = f. \end{cases}$$

Application of (SI) yields an $h: A \to C$ with $a_i h = b_i$ for all $i \in \mathbf{I}$.

Hence we have a $j \in \mathbf{I}$ with $h = \sigma(j)$ and the contradiction

$$a_j\sigma(j) = f \Leftrightarrow a_j\sigma(j) = g.$$

(ii) Form the $\mathbf{I}$-indexed discrete P-co-cone $(PB, x_i, A_i;\, i \in \mathbf{I})$ with $A_i = A$ and $\mathbf{I}$, x_i as before and proceed in an absolutely analogous way as above.

3.3. Proof of the theorem. (i) ⇒ (ii). Let $\xi: P \circ D \to \Delta X$ be a P-co-cone and consider the category $\tilde{\mathbf{D}}$ whose objects are all P-morphisms (x, B), such that there is a co-cone $\beta: D \to \Delta B$ with $(\Delta x)\xi = P \circ \beta$; because of 3.2 β is uniquely determined and denoted by $\beta_{x,B}$. A morphism $\bar{f}: (x, B) \to (y, C)$ is given by an $\mathbf{A}$-morphism $f: B \to C$ with $(Pf)x = y$. We have a canonical functor $\tilde{D}: \tilde{\mathbf{D}} \to \mathbf{A}$ and a P-cone $\tilde{\xi}: \tilde{\Delta}X \to P \circ \tilde{D}$ with $\tilde{\xi}(x, B) = x$ for all $(x, B) \in Ob\, \tilde{\mathbf{D}}$, which has a rigid P-semi-initial factorization $(\tilde{\Delta}e)(P \circ \tilde{\alpha}) = \tilde{\xi}$ with $e: X \to PA$ in $\mathbf{Q}$. Now, for every $d \in Ob\mathbf{D}$, we define $\tilde{\beta}_d(x, B) := \beta_{x,B}d$ for all $(x, B) \in Ob\tilde{\mathbf{D}}$, and get an $\mathbf{A}$-cone $\tilde{\beta}_d: \tilde{\Delta}Dd \to \tilde{D}$ because of 3.2. Furthermore, we have $\tilde{\xi}(\tilde{\Delta}\xi d) = P \circ \tilde{\beta}_d$, and applying (SI) we get a morphism $\alpha d: Dd \to A$ with $e(\xi d) = P\alpha d$. $\alpha: D \to \Delta A$ is a cone with $(\Delta e)\xi = P \circ \alpha$. It remains to show that this is a P-semi-final prolongation of ξ. Let y and β be as in (SF), that is $(y, B) \in Ob\tilde{\mathbf{D}}$ and $\beta = \beta_{y,B}$. With $t := \tilde{\alpha}(y, B)$ we then have $(Pt)e = y$ (and therefore $(\Delta t)\alpha = \beta$). Any other $s: A \to B$ with $(Ps)e = y$ leads to a morphism $\bar{s}: (e, A) \to (y, B)$ of $\tilde{\mathbf{D}}$ and hence to the equation $s\tilde{\alpha}(e, A) = \tilde{\alpha}(y, B) = t$. But the morphism $a := \tilde{\alpha}(e, A): A \to A$ fulfils the equations $(Pa)e = e$ and $\tilde{\alpha}(\Delta a) = \tilde{\alpha}$ and hence must be the identity morphism because of (R).

$$\begin{tikzcd}
 & & \tilde\Delta PA \arrow[dd, "P\circ\tilde\alpha"] \\
\tilde\Delta PDd \arrow[urr, dashed, "\tilde\Delta P\alpha d"] \arrow[r, "\tilde\Delta\xi d"] \arrow[drr, "P\circ\tilde\beta_d"'] & \tilde\Delta X \arrow[ur, "\tilde\Delta e"'] \arrow[dr, "\tilde\xi"] & \\
 & & P\circ\tilde{\mathbf{D}}
\end{tikzcd}$$

Fig. 5.

(ii) ⇒ (i). We proceed in an analogous way as before. Given a P-cone $\xi: \Delta X \to P \circ D$ we form the category $\tilde{\mathbf{D}}$, whose objects are P-co-morphisms (x, B), such that there is a cone $\beta = \beta_{x,B}: \Delta B \to D$ with $\xi(\Delta x) = P \circ \beta$. We consider a P-semi-final prolongation $(\tilde{\Delta}e)\tilde{\xi} = P \circ \tilde{\alpha}$ with $e: X \to PA$ in $\mathbf{Q}$ of the canonical P-co-cone $\tilde{\xi}: P \circ \tilde{D} \to \tilde{\Delta}X$. For every $d \in Ob\mathbf{D}$ we have an $\mathbf{A}$-cone $\tilde{\beta}_d: \tilde{D} \to \tilde{\Delta}Dd$ with $(\tilde{\Delta}\xi d)\tilde{\xi} = P \circ \tilde{\beta}_d$ and get a morphism $\alpha d: A \to Dd$ with $(P\alpha d)e = \xi d$. To prove that $(P \circ \alpha)(\Delta e) = \xi$ is a rigid P-semi-initial-factorization, let (y, B) be in $\tilde{\mathbf{D}}$. Then, with $b := \tilde{\alpha}(y, B)$ we have $ey = Pb$. (R) is fulfilled because of 2.2(2).

3.4. Remarks. (1) To prove the (i)-version of 3.2 we have not used condition (R) and the uniqueness part of (SI), which follows necessarily.

(2) Because of 3.2 (ii), statement 3.1 (ii) is equivalent to the corresponding statement about discrete P-co-cones.

(3) Note that the proof of 3.2 works without assuming that the respective categories have small hom-classes as Herrlich and Hoffmann did in the special case of $(\mathbf{E}, \mathbf{M})$-topological functors (cp. [12, 15]).

(4) Lemma 3.2 becomes wrong if one restricts conditions (i), (ii) resp. to small cones, co-cones resp. For a counter-example consider the functor $\mathbf{A} \to \mathbf{1}$, where $\mathbf{A}$ is any complete, co-complete resp. category being not a lattice.

4. Semi-topological functors

The generalized duality theorem leads to the following definition:

4.1. Definition. (1) $P: \mathbf{A} \to \mathbf{X}$ is called *semi-topological,* iff every P-cone has a rigid P-semi-initial lifting. Dual notion: *co-semi-topological.*

(2) P is called *(properly) topological,* iff every P-cone has a (proper) P-initial lifting. Dual notion: *(properly) co-topological.*

(3) P is called a *(proper) fibration,* iff every P-morphism has a (proper) P-initial lifting. Dual notion: *(proper) co-fibration.*

We have as immediate consequences of 3.1:

4.2. Corollary. *The following statements are equivalent:*

(i) *P is semi-topological.*

(ii) *Every (discrete) P-co-cone has a P-semi-final lifting.*

(iii) *Every (discrete) P-cone ξ has a P-semi-initial lifting* (*) *with* $(e, A) \in Epi(P)$.

(iv) *Every (discrete) P-cone ξ has a P-semi-initial lifting* (*) *with* $(e, A) \in Quot(P)$.

4.3. Corollary. *The following are equivalent:*

(i) *P is (properly) topological.*

(ii) *P is (properly) co-topological.*

(iii) *P is semi-topological and a (proper) fibration.*

(iv) *P is co-semi-topological and a (proper) co-fibration.*

Proof. It remains to show (iii) $\Rightarrow$ (i): One gets a (proper) P-initial lifting of a P-cone ξ taking first a P-semi-initial lifting (*) and then a (proper) P-initial lifting of e.

4.4. Examples (of semi-topological functors). (1) All topological functors are semi-topological, for instance the underlying **Ens**-functor of the categories of topological spaces, uniform spaces, proximity spaces, limit spaces, nearness spaces, measure spaces, Dynkin-systems, pre-ordered sets etc.

More generally, any $(\mathbf{E}, \mathbf{M})$-topological functor in the sense of Herrlich [12] is semi-topological.

(2) All monadic functors over **Ens** are semi-topological, for instance the underlying **Ens**-functor of the categories of semi-groups, monoids, groups, rings, R-modules, (associative) R-(Lie-; Jordan-) algebras, loops, graphs, compact T_2-spaces etc. More generally, any regular functor in the sense of Herrlich [13] is semi-topological.

From these two groups of examples one gets many new ones applying the following proposition. Moreover, in Sections 6 and 7 we shall give very general categorical methods to construct further examples (cp. 7.8).

4.5. Proposition. *Any functor $P:\mathbf{A}\to\mathbf{X}$ of the following is semi-topological*:

(a) *P is the embedding of an arbitrary full reflective subcategory.*

(b) *P is the composition of two semi-topological functors.*

(c) *P is the composition*

$$\mathbf{A}\xrightarrow{I}\mathbf{B}\xrightarrow{J}\mathbf{K}\xrightarrow{Q}\mathbf{X}$$

where Q is topological, J is a full epi-reflective and I a full co-reflective embedding with co-reflection map ρ, such that $Q\circ\rho$ is an isomorphism.

Proof. (a) Every P-cone factorizes over a reflection map and every $\mathbf{A}$-cone is P-initial, because P is full and faithful.

(b) Let $Q:\mathbf{K}\to\mathbf{X}$ and $R:\mathbf{A}\to\mathbf{K}$ be semi-topological with $P=Q\circ R$ and let $\xi:\Delta X\to P\circ D$ be a P-cone. Form a Q-semi-initial factorization $(Q\circ\kappa)(\Delta e)=\xi$ with a Q-epimorphism e and an R-semi-initial factorization $(R\circ\alpha)(\Delta k)=\kappa$ with an R-epimorphism k.

(c) Because of (b), a P-cone $\xi:\Delta X\to P\circ D$ has a $Q':=Q\circ J$-semi-initial factorization $(Q'\circ\beta)(\Delta e)=\xi$, where $e:X\to QB$ can be chosen as an epimorphism of $\mathbf{X}$. Then $\alpha:=\beta(\Delta\rho B)$ is an $\mathbf{A}$-cone and $(P\circ\alpha)\,\Delta(Q\rho B)^{-1}(\Delta e)=\xi$ is a P-semi-initial factorization with the P-epimorphism $(Q\rho B)^{-1}e$.

4.6. Remarks. (1) By 4.5 (a), (b) restrictions of topological functors to arbitrary full reflective subcategories are semi-topological. In 8.3 we shall show that – vice versa – every semi-topological functor has a presentation as a full reflective restriction of a topological functor.

(2) It is still an open problem whether or not a monadic functor over an arbitrary base-category is semi-topological. A solution of this problem would include a solution of the co-completeness problem for monadic functors. This can be seen from the following theorem which collects some of the internal lifting properties of semi-topological functors.

4.7. Theorem. *Let $P:\mathbf{A}\to\mathbf{X}$ be semi-topological. Then one has*:

(1) *P is faithful.*

(2) *P has a left adjoint.*

(3) *If X is ($\mathbf{D}$-) complete, so is $\mathbf{A}$.*

(4) *If $\mathbf{X}$ is ($\mathbf{D}$-) co-complete, so is $\mathbf{A}$.*

(5) *If $\mathbf{X}$ has a set of generators, so has $\mathbf{A}$.*

Proof. (1) Cp. 3.2.

(2) Apply (SF) in case of empty co-cones.

(3)(4) Cp. 2.4 and 2.8.

(5) This follows from (1) and (2).

Further (external) lifting properties of semi-topological functors are proved in [36].

5. Locally orthogonal Q-functors

In this section we try to get a finer description of the different kinds of P-semi-initial factorizations $(P \circ \mu)(\Delta q) = \xi$ of a P-cone ξ. The duality Theorem 3.1 suggests to consider only those factorizations where q belongs to a fixed subclass $\mathbf{Q} \subset Mor(\mathbf{P})$. For some considerations it is also convenient to fix a subclass $\mathbf{M} \subset Cone(\mathbf{A})$ and to demand $\mu \in \mathbf{M}$. In the following we shall always assume that $\mathbf{Q}(\mathbf{M})$ is closed under composition with $\mathbf{A}$-isomorphisms from the left (right), i.e. for all $p: X \to PA$ in $\mathbf{Q}$, all isomorphisms $i: A \to B$ of $\mathbf{A}$ and all $\mu: \Delta B \to D$ in $\mathbf{M}$ we have $(Pi)p$ in $\mathbf{Q}$ and $\mu(\Delta i)$ in $\mathbf{M}$.

5.1. Definition. A *locally orthogonal* $(\mathbf{Q}, \mathbf{M})$*-factorization* of a P-cone $\xi: \Delta Y \to P \circ D$ consists of a P-morphism $q: Y \to PB$ in $\mathbf{Q}$ and an $\mathbf{A}$-cone $\mu: \Delta B \to D$ in $\mathbf{M}$ with

$$(P \circ \mu)(\Delta q) = \xi,$$

such that the following "local diagonalization property" holds:

(LD) For all $p: X \to PA$ in $\mathbf{Q}$, all $\mathbf{X}$-morphisms $x: X \to Y$ and all $\mathbf{A}$-cones $\alpha: \Delta A \to D$ with $(P \circ \alpha)(\Delta p) = \xi(\Delta x)$ there exists a unique $t: A \to B$ with $(Pt)p = qx$ and $\mu(\Delta t) = \alpha$ (cp. Fig. 3).

$P: \mathbf{A} \to \mathbf{X}$ is called a *locally orthogonal* $(\mathbf{Q}, \mathbf{M})$*-functor*, iff $\mathbf{Q}$ contains the class $Iso(P)$ (cp. 2.6) and every P-cone admits a locally orthogonal $(\mathbf{Q}, \mathbf{M})$-factorization. Sometimes we replace the prefix $(\mathbf{Q}, \mathbf{M})$ by $\mathbf{Q}$, if the class $\mathbf{M}$ is not specified, i.e. $\mathbf{M} = Cone(\mathbf{A})$.

The following theorem establishes the connection between semi-topological and locally orthogonal $\mathbf{Q}$-functors.

5.2. Theorem. *The following assertions are equivalent*:

(i) *P is semi-topological.*

(ii) *P is a locally orthogonal Quot(P)-functor.*

(iii) *P is a locally orthogonal* $\mathbf{Q}$*-functor for some* $\mathbf{Q} \subset Epi(P)$.

(iv) *P is a locally orthogonal* $\mathbf{Q}$*-functor for some* $\mathbf{Q} \subset Mor(P)$.

Proof. (i) $\Rightarrow$ (ii) follows directly from 2.6, 2.9 and 4.2.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial.

(iv) $\Rightarrow$ (i). Obviously, locally orthogonal $\mathbf{Q}$-factorizations are rigid. So it remains to show that they are also P-semi-initial. But this follows directly from (LD) taking for p an identity morphism which belongs to $\mathbf{Q}$ because of the assumption $Iso(P) \subset \mathbf{Q}$.

5.3. Remarks. (1) $Quot(P)$ is the smallest subclass of $Mor(P)$, such that the semi-topological functor P is a locally orthogonal $\mathbf{Q}$-functor.

(2) If P is a locally orthogonal $\mathbf{Q}$-functor, then $\mathbf{Q}$ consists of all P-morphisms which do not admit a proper locally orthogonal $\mathbf{Q}$-factorization, i.e.

$$\mathbf{Q} = \{p : X \to PA \text{ in } Mor(P) \mid \text{If } (Pm)q = p \text{ is a locally orthogonal } \mathbf{Q}\text{-factorization, then } m \text{ is an isomorphism}\}.$$

Locally orthogonal $\mathbf{Q}$-functors can be described equivalently by a factorization structure of the category $\mathbf{A}$. This gives us an internal characterization of semi-topological functors.

5.4. Theorem. (1) *If P is a locally orthogonal* $(\mathbf{Q}, \mathbf{M})$*-functor, then we have*:

(a) *P is a faithful functor with a left-adjoint F such that the co-unit $\varepsilon : F \circ P \to \mathbf{A}$ belongs pointwise to*

$$\mathbf{E} := {}^{in}\mathbf{Q} := \{e : A \to B \text{ in } \mathbf{A} \mid (Pe, B) \in \mathbf{Q}\}.$$

(b) $\mathbf{A}$ *is a locally orthogonal* $(\mathbf{E}, \mathbf{M})$*-category* (*i.e., the identical functor on* $\mathbf{A}$ *is a locally orthogonal* $(\mathbf{E}, \mathbf{M})$*-functor*).

(2) *If* $\mathbf{E}$ *is a subclass of* $\mathbf{A}$ *such that conditions* (a) *and* (b) *hold, then P is a locally orthogonal* $(\mathbf{Q}, \mathbf{M})$*-functor with*

$$\mathbf{Q} := {}^{ex}\mathbf{E} := \{q : X \to PB \text{ in } Mor(P) \mid (\varepsilon B)(Fq) \in \mathbf{E}\}.$$

Proof. (1) (a) Let $(Pm)q = P\varepsilon A$ with $m : B \to A$ be a locally orthogonal $(\mathbf{Q}, \mathbf{M})$-factorization of $(P\varepsilon A, A)$ with $A \in Ob\ \mathbf{A}$ which is, in particular, a rigid P-semi-initial factorization. Then we have $(Pm)q(\eta PA) = PA$, where η is the unit of the adjunction, and there is a unique $t : A \to B$ with $q(\eta PA) = Pt$ and $mt = A$. Because of the rigidity it follows that also $tm = B$.

(b) If $\alpha : \Delta A \to D$ is an $\mathbf{A}$-cone, then $P \circ \alpha : \Delta PA \to P \circ D$ is a P-cone which has a locally orthogonal $(\mathbf{Q}, \mathbf{M})$-factorization $(P \circ \mu)\Delta q = P \circ \alpha$. The P-morphism $q : PA \to PB$ can be lifted to an $\mathbf{A}$-morphism $e : A \to B$ with $\mu(\Delta e) = \alpha$ and this is the desired locally orthogonal $(\mathbf{E}, \mathbf{M})$-factorization. Note that $\mathbf{E}$ is closed under composition with isomorphisms and contains $Iso(\mathbf{A})$.

(2) $\mathbf{Q}$ is closed under composition with isomorphisms and contains $Iso(P)$ because ε belongs pointwise to $\mathbf{E}$. A P-cone $\xi : \Delta X \to P \circ D$ induces an $\mathbf{A}$-cone $\alpha : \Delta A \to D$ with a locally orthogonal $(\mathbf{E}, \mathbf{M})$-factorization $\mu(\Delta e) = \alpha$. Then $(P \circ \mu)(\Delta Pe) \cdot (\Delta \eta X) = \xi$ is the desired locally orthogonal $(\mathbf{Q}, \mathbf{M})$-factorization.

5.5. Remarks. (1) Because of 5.4 it would be sufficient to consider only $(\mathbf{E}, \mathbf{M})$-structures of $\mathbf{A}$, since every $\mathbf{Q}$ has an "internalization" ${}^{in}\mathbf{Q}$ and every $\mathbf{E}$ has an "externalization" ${}^{ex}\mathbf{E}$. But this seems to be not very natural, if one thinks of examples, and, besides, gives no simplification of proofs.

(2) The condition of faithfulness in 5.4(1)(a) can be omitted: cp. 6.4 below.

At the end of this section we briefly discuss the question under which conditions the class $\mathbf{M}$ can be chosen as a subclass of

$$Mono(\mathbf{A}) := \{\mu : \Delta B \to D \text{ in } Cone(\mathbf{A}) \mid \mu \text{ is a mono-cone}\}.$$

Recall that $\mu : \Delta B \to D$ is a *mono-cone*, iff for all $f, g : A \to B$ in $\mathbf{A}$ with $\mu(\Delta f) = \mu(\Delta g)$ it follows $f = g$.

5.6. Proposition. *The following two assertions for* $\mathbf{Q} \subset Mor(P)$ *are equivalent*:

(i) a. *P is a locally orthogonal $\mathbf{Q}$-functor.*

b. *$\mathbf{A}$ has co-equalizers.*

c. *$\mathbf{Q}$ is closed under composition with extremal epimorphisms of $\mathbf{A}$ from the left.*

(ii) *P is a locally orthogonal $(\mathbf{Q}, Mono(\mathbf{A}))$-functor.*

Proof. (i) $\Rightarrow$ (ii). Let $(P \circ \mu)(\Delta q) = \xi$ be a locally orthogonal $\mathbf{Q}$-factorization and assume $\mu(\Delta f) = \mu(\Delta g)$. Then μ has a factorization over the co-equalizer $c : B \to C$ of f and g with an $\mathbf{A}$-cone $\gamma : \Delta C \to D$. Because of condition c. we get from (LD) a morphism $t : C \to B$ with $(Ptc)q = q$ and $\mu(\Delta t) = \gamma$, hence $tc = B$ and $f = g$.

(ii) $\Rightarrow$ (i) b. Given $f, g : A \to B$ we consider the discrete $\mathbf{A}$-cone of all morphisms $c_i : B \to C_i$ with $c_i f = c_i g$, which has a $({}^{in}\mathbf{Q}, Mono(\mathbf{A}))$-factorization $m_i c = c_i$, $i \in \mathbf{I}$. Then c turns out to be a co-equalizer of f and g. Here we use the fact that $\mathbf{Q}$ consists of P-epimorphisms only; this is shown in Lemma 6.4 below.

c. Apply 5.3(2).

6. Characterization and existence theorems

Generalizing a construction of Herrlich [12] we associate to every functor $P : \mathbf{A} \to \mathbf{X}$ and each subclass $\mathbf{Q} \subset Mor(P)$ the following functor $Q : \mathbf{K} := \mathbf{K}_{P,\mathbf{Q}} \to \mathbf{X}$: Objects of $\mathbf{K}$ are the elements of $\mathbf{Q}$, a $\mathbf{K}$-morphism $[x, f] : (p, A) \to (q, B)$ consists of an $\mathbf{X}$-morphism x and an $\mathbf{A}$-morphism $f : A \to B$ with $qx = (Pf)p$ (cp. Fig. 6), and Q is the obvious projection functor.

To formulate our characterization theorem for locally orthogonal $\mathbf{Q}$-functors we need the following notations.

6.1. Definition. P is called $\mathbf{Q}$-*co-complete*, iff the following two conditions are satisfied:

(a) For every $p:X\to PA$ in $\mathbf{Q}$ and every $x:X\to Y$ in $\mathbf{X}$ there exists a $q:Y\to PB$ in $\mathbf{Q}$ and an $f:A\to B$ in $\mathbf{A}$ such that Fig. 6 is a "*semi-P-push-out*", that is $qx=(Pf)p$, and for all P-morphisms $y:Y\to PC$ and $\mathbf{A}$-morphisms $g:A\to C$ with $yx=(Pg)p$ there is a unique $h:B\to C$ with $(Ph)q=y$ and $hf=g$.

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\ PA \arrow[r, "Pf"'] & PB \end{tikzcd}$$

Fig. 6.

(b) For each discrete P-cone $(q_i:Y\to PB_i;\ i\in\mathbf{I})$ with q_i in $\mathbf{Q}$ for all $i\in\mathbf{I}$ there exists an $\mathbf{A}$-co-cone $(e_i:B_i\to B;\ i\in\mathbf{I})$ and a $q:Y\to PB$ in $\mathbf{Q}$ such that Fig. 7 is a "*multiple P-push-out*", that is $(Pe_i)q_i=q$ for all $i\in\mathbf{I}$, and for each $\mathbf{A}$-co-cone $(g_i:B_i\to C;\ i\in\mathbf{I})$ and all P-morphisms $y:Y\to PC$ with $(Pg_i)q_i=y$ there is a unique $g:B\to C$ with $ge_i=g_i$ for all $i\in\mathbf{I}$ and $(Pg)q=y$.

$$\begin{tikzcd} Y \arrow[d, "q_i"'] \arrow[dr, "q"] & \\ PB_i \arrow[r, "Pe_i"'] & PB \end{tikzcd}$$

Fig. 7.

6.2. Remarks. (1) If P is semi-topological, the semi-P-push-outs and multiple P-push-outs can be constructed from ordinary push-outs of $\mathbf{X}$. If one has p and x like in Fig. 6, at first form a push-out (p',x') in $\mathbf{X}$ and then take a P-semi-final prolongation of the P-co-morphism $x':PA\to Y$. Analogously, given P-morphisms q_i like in Fig. 7, form a co-intersection in $\mathbf{X}$ and take a P-semi-final prolongation of the arisen P-co-cone.

(2) Let $p:X\to PA$ be in $Mor(P)$ and define a discrete P-cone $(p_i:X\to PA_i;\ i\in\mathbf{I})$ with $\mathbf{I}:=Mor(\mathbf{A})$ by $p_i:=p$ and $A_i:=A$ for all $i\in\mathbf{I}$. Then a multiple P-push-out of $(p_i:X\to PA_i;\ i\in\mathbf{I})$ exists if and only if $p:X\to PA$ is a P-epimorphism. The "only if" part is trivial. Let $e_i:A_i\to B$ and $q:X\to PB$ form a multiple P-push-out and assume $(Pa)p=(Pb)p$ with two different morphisms $a,b:A\to C$. The class

$$\mathbf{K}:=\{g:B\to C\mid ge_i\in\{a,b\}\text{ for all } i\in\mathbf{I}\}$$

is not empty. Hence there is a surjection $\sigma:\mathbf{I}\to\mathbf{K}$. For

$$g_i:=\begin{cases} a & \text{in case } \sigma(i)e_i=b\\ b & \text{in case } \sigma(i)e_i=a\end{cases}$$

one gets a morphism g with $ge_i = g_i$ for all $i \in \mathbf{I}$ and a $j \in \mathbf{I}$ with $\sigma(j) = g$. Because of

$$\sigma(j)e_i = a \Leftrightarrow \sigma(j)e_i = b_i$$

this is impossible.

6.3. Theorem. *Let* $\mathbf{Q}$ *be a subclass of* $Mor(P)$ *containing* $Iso(P)$. *The following assertions are equivalent*:

(i) P *is a locally orthogonal* $\mathbf{Q}$*-functor.*

(ii) $Q: \mathbf{K}_{P,\mathbf{Q}} \to \mathbf{X}$ *is properly topological.*

(iii) $Q: \mathbf{K}_{P,\mathbf{Q}} \to \mathbf{X}$ *is a proper fibration and co-fibration and its fibres are large complete lattices.*

(iv) P *is* $\mathbf{Q}$*-co-complete.*

Proof. (i) $\Rightarrow$ (ii). Let $R: \mathbf{K} \to \mathbf{A}$ be the second projection functor. We have a natural transformation $\sigma: Q \to P \circ R$ with $\sigma(p, A) = p$ for all (p, A) in $\mathbf{Q}$. Now, let $\xi: \Delta X \to Q \circ D$ be a Q-cone. Then the P-cone $(\sigma \circ D)\xi: \Delta X \to P \circ R \circ D$ has a locally orthogonal $\mathbf{Q}$-factorization

$$(P \circ \mu)(\Delta p) = (\sigma \circ D)\xi.$$

ξ and μ form a $\mathbf{K}$-cone $\Delta(p, A) \to D$ which Q-lifts ξ.

$$\begin{tikzcd}
\Delta X \arrow[r, "\xi"] \arrow[d, "\Delta p"'] & Q \circ D \arrow[d, "\sigma \circ D"] \\
\Delta PA \arrow[r, "P \circ \mu"'] & P \circ R \circ D
\end{tikzcd}$$

Fig. 8.

So it remains to show, that the $\mathbf{K}$-cone given by Fig. 8 is Q-initial. But this can easily be checked by using (LD).

(ii) $\Rightarrow$ (iii) is well-known and proved straightforward for every topological functor.

(iii) $\Rightarrow$ (iv). The meaning of 6.1 (a) is just that Q is a proper co-fibration. Now let $q_i: Y \to PB_i$ be given as in 6.1 (b) and form its direct sum in $Q^{-1}Y$. Then we have the situation of Fig. 7 and it is still to be shown that this is a multiple P-push-out. Let $g_i: B_i \to C$ and $y: Y \to PC = Q(PC, C)$ be as in 6.1 (b). There exist $q': Y \to PB'$ in $\mathbf{Q}$ and $f: B' \to C$ in $\mathbf{A}$ such that $[y, f]: (q', B') \to (PC, C)$ is a Q-initial morphism of $\mathbf{K}$. Hence there are uniquely determined morphisms $h_i: B_i \to B'$ with $Ph_iq_i = q'$ and $fh_i = g_i$, $i \in \mathbf{I}$. Now in $Q^{-1}Y$ we have $(q_i, B_i) \leq (q', B')$, hence $(q, B) \leq (q', B')$. Composition with f yields the desired morphism $g: B \to C$ with $ge_i = g_i$, $i \in \mathbf{I}$.

(iv) $\Rightarrow$ (i). Given a P-cone $\xi: \Delta Y \to P \circ D$ we consider all P-morphisms $q_i: Y \to PB_i$ such that there is an $\mathbf{A}$-cone $\mu_i: \Delta B_i \to D$ with $(P \circ \mu_i)(\Delta q_i) = \xi$. We form a multiple

P-push-out $(Pe_i)q_i = q$ in $\mathbf{Q}$ and get a unique $\mathbf{A}$-cone $\mu : \Delta B \to D$ with $\mu(\Delta e_i) = \mu_i$, $i \in \mathbf{I}$, and $(P \circ \mu)(\Delta q) = \xi$. Now we consider the commutative square of Fig. 3, form a semi-P-push-out $p'x = (Pf)p$ with $p' : Y \to PC$ in $\mathbf{Q}$ and get a unique $\mathbf{A}$-cone $\alpha' : \Delta C \to D$ with $\alpha'(\Delta f) = \alpha$ and $(P \circ \alpha')\Delta p' = \xi$. Hence there must be an index i with $p' = q_i$. The morphism $t := e_i f$ is a suitable diagonal since p is P-epimorphic because of 6.2(2).

6.4. Corollary. *If P is a locally orthogonal $\mathbf{Q}$-functor, then $\mathbf{Q} \subset Epi(P)$. If $\mathbf{A}$ is a locally orthogonal $\mathbf{E}$-category, then $\mathbf{E} \subset Epi(\mathbf{A})$.*

Hence in the following we can restrict ourselves to subclasses $\mathbf{Q}$ of P-epimorphisms and $\mathbf{E}$ of $\mathbf{A}$-epimorphisms.

6.5. Corollary. *Let $\mathbf{E}$ contain all isomorphisms of $\mathbf{A}$.*

(1) *The following assertions are equivalent:*

(i) *$\mathbf{A}$ is a locally orthogonal $\mathbf{E}$-category.*

(ii) *$\mathbf{A}$ is $\mathbf{E}$-co-complete, i.e. a push-out of an $\mathbf{E}$-morphism exists and belongs to $\mathbf{E}$ and arbitrary co-intersections of $\mathbf{E}$-morphisms exist and belong to $\mathbf{E}$.*

(2) *The following two assertions are equivalent:*

(i) *$\mathbf{A}$ is a locally orthogonal $(\mathbf{E}, Mono(\mathbf{A}))$-category.*

(ii) (a) *$\mathbf{A}$ is $\mathbf{E}$-co-complete.*

(b) *$\mathbf{A}$ has co-equalizers.*

(c) *$\mathbf{E}$ is closed under composition with extremal epimorphisms from the left.*

Connecting 6.5 with 5.4 leads to the following existence theorem for locally orthogonal $\mathbf{Q}$-functors.

6.6. Theorem. *Let $\mathbf{E}$ contain all isomorphisms of $\mathbf{A}$. If $\mathbf{A}$ is $\mathbf{E}$-co-complete and $P: \mathbf{A} \to \mathbf{X}$ is a functor with a left adjoint such that the co-unit belongs pointwise to $\mathbf{E}$, then P is a locally orthogonal ${}^{ex}\mathbf{E}$-functor. If, moreover, $\mathbf{A}$ has co-equalizers and if $\mathbf{E}$ is closed under composition with extremal epimorphisms from the left, then P is a locally orthogonal $({}^{ex}\mathbf{E}, Mono(\mathbf{A}))$-functor.*

6.7. Corollary. *Let $\mathbf{E}$ consist of all (regular; extremal) epimorphisms of $\mathbf{A}$ and let $\mathbf{A}$ be (small) co-complete and $\mathbf{E}$-co-well-powered. Let P have a left adjoint such that the co-unit belongs pointwise to $\mathbf{E}$. Then P is a locally orthogonal ${}^{ex}\mathbf{E}$-functor, in particular, P is semi-topological.*

Note that the co-units of a faithful right adjoint functor are necessarily epimorphic. Hence we get from 4.7 and 6.7:

6.8. Corollary. *Let $\mathbf{X}$ be co-complete and $\mathbf{A}$ be co-well-powered. Then P is semitopological if and only if P is a faithful right adjoint functor and $\mathbf{A}$ is co-complete.*

7. Orthogonal Q-functors

In [33] we have considered the following Galois-correspondence between subclasses $\mathbf{Q}$ of $Mor(P)$ and $\mathbf{M}$ of $Cone(\mathbf{A})$, which are closed under composition with isomorphisms of $\mathbf{A}$ from the left (right).

$$\mathbf{Q}^{\perp}(P) := \{(B, \mu, D) \in Cone(\mathbf{A}) \mid (p, A) \perp (B, \mu, D) \text{ for all } (p, A) \in \mathbf{Q}\},$$

$$\mathbf{M}^{\perp}(P) := \{(p, A) \in Mor(P) \mid (p, A) \perp (B, \mu, D) \text{ for all } (B, \mu, D) \in \mathbf{M}\},$$

where $(p, A) \perp (B, \mu, D)$ means that for all P-morphisms $u: X \to PB$ and all $\mathbf{A}$-cones $\alpha: \Delta A \to D$ with $(P \circ \alpha)(\Delta p) = (P \circ \mu)(\Delta u)$ there is a unique $t: A \to B$ with $(Pt)p = u$ and $\mu(\Delta t) = \alpha$.

$$\begin{tikzcd} \Delta X \arrow[r, "\Delta p"] \arrow[d, "\Delta u"'] & \Delta PA \arrow[d, "P\circ\alpha"] \arrow[dl, dashed, "\Delta Pt"] \\ \Delta PB \arrow[r, "P\circ\mu"'] & P\circ D \end{tikzcd}$$

Fig. 9.

7.1. Definition. $P: \mathbf{A} \to \mathbf{X}$ is called an *orthogonal* $\mathbf{Q}$-*functor*, iff $\mathbf{Q}$ contains $Iso(P)$ and every P-cone $\xi: \Delta Y \to P \circ D$ admits a factorization

$$(P \circ \mu)(\Delta q) = \xi$$

with $q: Y \to PB$ in $\mathbf{Q}$ and $\mu: \Delta B \to D$ in $\mathbf{Q}^{\perp}(P)$.

7.2. Remark. With $\mathbf{M} := \mathbf{Q}^{\perp}(P)$ we have $\mathbf{M}^{\perp}(P) = \mathbf{Q}$ and $\mathbf{M}$ consists of P-initial cones only. Hence an orthogonal $\mathbf{Q}$-functor is an orthogonal $\mathbf{M}$-functor in the sense of [32]. Vice versa, every orthogonal $\mathbf{M}$-functor in the sense of [32] is an orthogonal $\mathbf{Q}$-functor with $\mathbf{Q} := \mathbf{M}^{\perp}(P)$. In this case we also speak of an *orthogonal* $(\mathbf{Q}, \mathbf{M})$-*functor.*

Clearly every orthogonal $\mathbf{Q}$-functor is a locally orthogonal $\mathbf{Q}$-functor. We try to prove some sort of converse proposition and consider at first the internal case.

7.3. Lemma. *For* $\mathbf{E} \subset \mathbf{A}$ *the following conditions are equivalent.*

(i) $\mathbf{A}$ *is an orthogonal* $\mathbf{E}$-*category.*

(ii) $\mathbf{A}$ *is a locally orthogonal* $\mathbf{E}$-*category and* $\mathbf{E}$ *is closed under composition.*

Proof. (i) $\Rightarrow$ (ii) is clear because of $\mathbf{E} = (\mathbf{E}^{\perp}(\mathbf{A}))^{\perp}(\mathbf{A})$.

(ii) $\Rightarrow$ (i). It suffices to show that a locally orthogonal $\mathbf{E}$-factorization $\mu(\Delta e) = \gamma$ of an $\mathbf{A}$-cone $\gamma: \Delta C \to D$ is already orthogonal, i.e. $\mu: \Delta B \to D$ belongs to $\mathbf{E}^{\perp}(\mathbf{A})$. μ has again a locally orthogonal $\mathbf{Q}$-factorization $\mu = \nu(\Delta p)$. Because of $pe \in \mathbf{E}$ and (LD) one gets a t with $tpe = e$. Since $\mathbf{E} \subset Epi(\mathbf{A})$ (cp. 6.4) p must be an isomorphism and $\mu = \nu(\Delta p)$ is already an orthogonal $\mathbf{Q}$-factorization, i.e. $\mu \cong \nu \in \mathbf{E}^{\perp}(\mathbf{A})$.

7.4. Theorem. *Let $\mathbf{A}$ be a locally orthogonal $\mathbf{E}$-category, let $\mathbf{E}$ be closed under composition and let P have a left adjoint functor such that the co-unit belongs pointwise to $\mathbf{E}$. Then P is an orthogonal ${}^{ex}\mathbf{E}$-functor.*

Proof. Because of 7.3 $\mathbf{A}$ is an orthogonal $\mathbf{E}$-category. The rest can be shown analogously to 5.4.

Via 7.4 one immediately gets existence theorems for orthogonal ${}^{ex}\mathbf{E}$-functors from those for locally orthogonal ${}^{ex}\mathbf{E}$-functors (cp. 6.6). Here we prove another existence theorem which generalizes the procedure described in 2.3(2) in case of groups (cp. also [32, Lemma 3.4]).

7.5. Theorem. *Let $\mathbf{A}$ be complete, well-powered and co-well-powered and let P be faithful and right-adjoint. Then there exists a class $\mathbf{Q} \subset Epi(P)$ such that P is an orthogonal $\mathbf{Q}$-functor with $\mathbf{Q}^{\perp}(P) \subset Mono(\mathbf{A})$.*

Proof. $\xi : \Delta X \to P \circ D$ induces an $\mathbf{A}$-cone $\alpha : \Delta FX \to D$ and it suffices to show that α admits a factorization $\alpha = (P \circ \mu)(\Delta e)$, where μ belongs to the class $\mathbf{M}$ of all P-initial mono-cones of $\mathbf{A}$ and $e : FX \to A$ is an epimorphism in $\mathbf{M}^{\perp}(\mathbf{A})$. Because pull-backs and intersections of $\mathbf{M}$'s are in $\mathbf{M}$ and $\mathbf{M}$ contains all equalizers this can be shown at first for morphisms using the usual factorization technique of Isbell–Kennison–Herrlich. Then small cones α are regarded as morphisms $a : FX \to \prod_d Dd$. Finally in the arbitrary case one has to factorize at first all $\alpha d = m_d e_d$ and then to take a representative small cone $(e_i, i \in \mathbf{I} \subset \mathbf{D})$ which can be factorized as described above.

7.6. Remark. Note that we have used only that $\mathbf{A}$ is co-well-powered with respect to the class $\mathbf{M}^{\perp}(\mathbf{A})$. On the other hand one can weaken the assumption of well-poweredness taking for $\mathbf{M}$ the class $(Epi(\mathbf{A}))^{\perp}(\mathbf{A}) \cap Init(P)$ instead of $Mono(\mathbf{A}) \cap Init(P)$, where $Init(P)$ denotes the class of P-initial cones. Then one only uses well-poweredness with respect to extremal monomorphisms. So an analysis of the proof of 7.5 yields, in case $P = \mathbf{A}$, the following consequence of 6.5:

7.7. Corollary. *Let $\mathbf{A}$ be complete and extremally well-powered and co-well-powered (well-powered and extremally co-well-powered). Then $\mathbf{A}$ has co-equalizers, push-outs of (extremal) epimorphisms and co-intersections of (extremal) epimorphisms.*

7.8. Examples (of orthogonal $\mathbf{Q}$-functors). (1) All examples of semi-topological functors given in 4.4 are already orthogonal $\mathbf{Q}$-functors (for a certain $\mathbf{Q}$).

(2) Let $\mathbf{A}$ be a well-bounded category with a generator $\mathbf{G}$ in the sense of Wischnewsky [39] (i.e. $\mathbf{A}$ fulfils the conditions of the "Continuous Functor Theorem" of Freyd and Kelly [7]). Then

$$(\mathbf{A}(G,-))_{\mathbf{G}} : \mathbf{A} \to \mathbf{Ens}^{\mathbf{G}}$$

is an orthogonal $\mathbf{Q}$-functor because of 7.4 (cp. also [33, Theorem (22)]). In particular, every locally presentable category in the sense of Gabriel and Ulmer [8] is the domain of a semi-topological functor over some power of $\mathbf{Ens}$.

(3) Let $\mathbf{K}$ be one of the categories of pointed metric spaces with metric-decreasing maps, of normed vector spaces or of Banach spaces with norm decreasing linear maps. The "unit-ball functor"

$$\bigcirc : \mathbf{K} \to \mathbf{Ens}$$

is an orthogonal ${}^{ex}\mathbf{E}$-functor where $\mathbf{E}$ contains the surjective morphisms of $\mathbf{K}$.

(4) The category $\mathbf{Cat}$ of small categories is a locally orthogonal $\mathbf{E}$-category where $\mathbf{E}$ is the class of all regular epimorphisms (cp. 6.6) but not an orthogonal $\mathbf{E}$-category because $\mathbf{E}$ is not closed under composition.

8. Topological completion

In this section we derive some important consequences from the characterization Theorem 6.3. As an immediate corollary we get Wyler's self-dual description of topological functors [40]:

8.1. Corollary. *P is topological if and only if P is a fibration and co-fibration and all fibres are large complete lattices.*

Proof. Apply 6.3 in case $\mathbf{Q} = Iso(P)$. Then Q coincides with P up to an equivalence.

For arbitrary P and $\mathbf{Q}$ with $\mathbf{Iso}(P) \subset \mathbf{Q}$ one has a full reflective embedding $E : \mathbf{A} \to \mathbf{K}_{P\mathbf{Q}}$ with $Q \circ E = P$, i.e. $EA = (PA, A)$ for all $A \in \mathbf{A}$. The reflector is just the second projection functor $R : \mathbf{K} \to \mathbf{A}$ with $R \circ E = \mathbf{A}$. The reflection morphism ρ fulfils the equation $Q \circ \rho = \sigma$ (cp. 6.3 (i)$\Rightarrow$(ii)). Hence we have proved:

8.2. Proposition. *Every locally orthogonal $\mathbf{Q}$-functor $P : \mathbf{A} \to \mathbf{X}$ admits a factorization $P = Q \circ E$ such that the following hold:*

(a) *$Q : \mathbf{K} \to \mathbf{X}$ is properly topological.*

(b) *$E : \mathbf{A} \to \mathbf{K}$ is a full reflective embedding with reflection map ρ, such that $(Q\rho K, RK) \in \mathbf{Q}$ for all $K \in Ob\, \mathbf{K}$.*

Because of 5.2 and 4.5 we get from 8.2 a representation theorem for semi-topological functors.

8.3. Theorem. *P is semi-topological if and only if P is a full reflective restriction of a topological functor.*

8.4. Remarks. (1) Full reflective embeddings are monadic. Hence every semi-topological functor admits a factorization over an "algebraic" and a topological one. So Hong's [19] notion of a "topologically algebraic" functor – which, in particular, is semi-topological – gets an additional justification by 8.3.

(2) In [35] it is shown that the factorization "full reflective embedding-topological functor" is a canonical one in the sense of Isbell's diagonalization property. This property still holds if "topological functor" is replaced by "fibration" and leads to an external characterization of topological functors and fibrations.

(3) Combining 8.3 with 6.6 or 6.8 yields the main result of [17].

9. Algebraic completion

In 8.3 we have represented any semi-topological functor as a composition of a monadic one followed by a topological one. Now we try to get a factorization over a topological one followed by a monadic one.

9.1. Lemma. *Let $P:\mathbf{A}\to\mathbf{X}$ be a locally orthogonal $\mathbf{Q}$-functor and let $K:\mathbf{A}\to\mathbf{X}^T$ be the comparison functor into the Eilenberg–Moore-category of P (T is the induced monad). Then K is a locally orthogonal $\mathbf{Q}^T$-functor with $\mathbf{Q}^T=\{(\bar{q},B)\in Mor(K)\,|\,(q,B)\in\mathbf{Q}\}$.*

Proof. Every K-cone $\bar{\xi}:\Delta(Y,y)\to K\circ D$ admits a locally orthogonal $\mathbf{Q}$-factorization $(P\circ\mu)(\Delta q)=\xi$ of its underlying P-cone $\xi:\Delta Y\to P\circ D$. It suffices to show that $q:Y\to PB$ induces an $\mathbf{X}^T$-morphism $\bar{q}:(Y,y)\to KB=(PB,P\varepsilon B)$. Because of

$$(P\circ\mu)(\Delta qy)=(P\circ\varepsilon\circ D)(P\circ F\circ\xi)$$

there exists an $e:FY\to B$ with $Pe=qy$. Hence we have $q=(Pe)(\eta Y)$ and then $e=(\varepsilon B)(Fq)$ and $qy=(P\varepsilon B)(PFq)$.

9.2. Theorem. *Let $\mathbf{X}$ be an orthogonal $(\mathbf{E},\mathbf{M})$-category. Consider the following two assertions for $P:\mathbf{A}\to\mathbf{X}$:*

(i) *There exists a pair $(\mathbf{E}_P,\mathbf{M}_P)$ such that*

(a) *$\mathbf{A}$ is an orthogonal $(\mathbf{E}_P,\mathbf{M}_P)$-category.*

(b) *$P\mathbf{E}_P\subset\mathbf{E}$, $P\mathbf{M}_P\subset\mathbf{M}$.*

(c) *P has a left adjoint such that the co-unit belongs pointwise to $\mathbf{E}_P$.*

(ii) *P has a factorization $P=U\circ Q\circ E$ such that*

(a) *$U:\mathbf{Y}\to\mathbf{X}$ is monadic and the induced monad functor preserves $\mathbf{E}$'s.*

(b) *$Q:\mathbf{K}\to\mathbf{Y}$ is (properly) topological.*

(c) *$E:\mathbf{A}\to\mathbf{K}$ is a full epi-reflective embedding with reflection map ρ such that $U\circ Q\circ\rho$ belongs pointwise to $\mathbf{E}$.*

Then (i) $\Rightarrow$ (ii) *while* (ii) $\Rightarrow$ (i) *in case* $\mathbf{M}\subset Mono(\mathbf{X})$.

Proof. (i)$\Rightarrow$(ii). Take U to be the underlying functor of the Eilenberg–Moore-category of P. P is a locally orthogonal $\mathbf{Q}$-functor with $\mathbf{Q}={}^{ex}\mathbf{E}_P$ (cp. 5.4). Let $K=E\circ Q$ be the factorization 8.2. $U\circ Q\circ \rho$ belongs pointwise to $\mathbf{E}$. In the proof of 9.1, e is in $\mathbf{E}_P$ and $Pe=qy$ in $\mathbf{E}$; hence $q\in\mathbf{E}$ because of $y\in Epi(\mathbf{X})$. It remains to show that $T=P\circ F$ preserves $\mathbf{E}$'s. If $Fe=mp$ is an orthogonal $(\mathbf{E}_P,\mathbf{M}_P)$-factorization with $e:X\to Y$ in $\mathbf{E}$, then there is a unique s with $se=(Pp)(\eta X)$ and $(Pm)s=\eta Y$. Furthermore one has a morphism t with $(Pt)(\eta Y)=s$ which turns out to be the inverse of m. Hence $Fe\in\mathbf{E}_P$ and $Te\in\mathbf{E}$.

(ii)$\Rightarrow$(i). The $(\mathbf{E},\mathbf{M})$-factorization structure of $\mathbf{X}$ can be lifted in three steps along U, Q and E: First U lifts $(\mathbf{E},\mathbf{M})$ to $(\mathbf{E}_U,\mathbf{M}_U):=(U^{-1}\mathbf{E},U^{-1}\mathbf{M})$ because of $T\mathbf{E}\subset\mathbf{E}$. Note that $\mathbf{M}_U$ consists of U-initial cones only because of $\mathbf{M}\subset Mono(\mathbf{X})$. Q lifts $(\mathbf{E}_U,\mathbf{M}_U)$ to $(\mathbf{E}',\mathbf{M}')$ where $\mathbf{E}':=Q^{-1}\mathbf{E}_U$ and $\mathbf{M}'$ consists of all Q-initial cones of $Q^{-1}\mathbf{M}_U$. Finally E lifts $(\mathbf{E}',\mathbf{M}')$ to $(\mathbf{E}_P,\mathbf{M}_P):=(E^{-1}\mathbf{E}',E^{-1}\mathbf{M}')$ because the reflection map belongs pointwise to $\mathbf{E}'$. The co-unit of P belongs pointwise to $\mathbf{E}_P$ because $\mathbf{M}_P$ consists of P-initial cones only.

9.3. Remark. If $\mathbf{X}$ is an orthogonal $(\mathbf{E},\mathbf{M})$-category, the comparison functor $K:\mathbf{A}\to\mathbf{Y}=\mathbf{X}^T$ is $(\mathbf{E}_U,\mathbf{M}_U)$-topological in the sense of Herrlich [12], because $\mathbf{M}_U$-cones have K-initial liftings. (In the proof 9.1 take $\bar{\xi}$ to be in $\mathbf{M}_U$; then q belongs to $\mathbf{E}\cap\mathbf{M}=Iso(\mathbf{X})$.) So under the hypothesis 9.2 (i) we have constructed a factorization of P over a relatively topological functor followed by a monadic one. This is the typical representation of an underlying **Ens**-functor of a category of topological algebras.

Note added in proof

(1) Problem 4.6(2) is solved by J. Adámek (Bull. Austral. Math. Soc. 17 (1977) 433–450).

(2) The relationship between semi-topological functors, topologically algebraic functors, and orthogonal $\mathbf{Q}$-functors is independently investigated in forthcoming papers by R. Börger and the author (Cahiers Topologie Geóm. Différentielle) and by H. Herrlich, R. Nakagawa, G. E. Strecker, and T. Titcomb (Canad. J. Math.).

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